The Stone–Čech Compactification

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In this paper we will be discussing the Stone–Čech compactification.

To begin we will need to understand a few topics in topology, starting with special types of spaces. For example we must define what a Hausdorff space is.

Definition (Hausdorff Space)**.** Let X be a topological space. Suppose that x_1 and x_2 are any two distinct points in X. We say that X is a **Hausdorff space** when there is a neighborhood U_1 of x_1 and a neighborhood U_2 of x_2 such that U_1 and U_2 are disjoint. That is, any two points in a Hausdorff space may be separated or "housed off" from each other.

We may consider a Hausdorff space visually as seen in Figure 1.

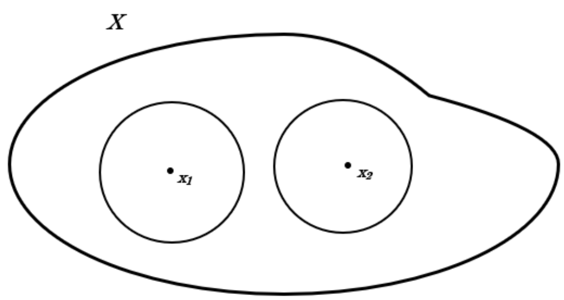

Figure 1: An example of a Hausdorff space X, with the example of two disjoint neighborhoods surrounding the points x_1 and x_2

Hausdorff spaces are fairly common, in fact if (X, d) is a metric space then (X, d) is a Hausdorff space as well. From this we can conclude that sets such as $\mathbb{R}$ are Hausdorff spaces. Another important type of space is called a completely regular space.

Definition (Completely Regular)**.** Let X be a space and let all singletons be closed in X. We say that X is **completely regular** if for any point $x_0 \in X$ and every closed set $A \subset X$ where $x_0 \notin A$ we have a continuous function $f : X \to [0, 1]$ such that

$$f(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \in A \end{cases}$$

We will be considering the behaviors of mappings between such spaces, especially those of a special type of mapping known as an imbedding.

Definition (Imbedding)**.** Let X and Y be topological spaces and define a continuous one-to-one mapping $f : X \to Y$. Define $Z \subset Y$ as the image of X under f. Then we may use Z to define a new bijective mapping $g : X \to Z$. If g is a homeomorphism then f is an **imbedding** of X in Y.

This will not be the last time that the concept of a homeomorphism will be relevant in our discussion, and so we will provide the following definition for the purposes of clarity and to provide a comprehensive preliminary review.

Definition (Homeomorphism)**.** Let X and Y be two topological spaces with the mapping $f : X \to Y$ between them. If f is a continuous bijection and the function $f^{-1} : Y \to X$ is continuous as well we call f a **homeomorphism**.

There is a preliminary theorem we must also cover known as the Imbedding Theorem

Theorem 1 (Imbedding Theorem)**.** Let X be a space with the property that all singleton sets are closed. Let $\{f_\alpha\}_{\alpha\in J}$ be a set of continuous functions where $f_\alpha : X \to \mathbb{R}$ with index set J. Furthermore, the family of functions has the property that for any $x_0 \in X$ with any neighborhood U of x_0 it is true that there exists some $\alpha \in J$ such that $f_\alpha(x) = 0$ when $x \notin U$ and $f_\alpha(x_0) > 0$. Then there is a function $F : X \to \mathbb{R}^J$ with

$$F(x) = (f_\alpha(x))_{\alpha\in J}$$

is an imbedding of X in $\mathbb{R}^J$. In addition, for each α, f_α maps X into $[0, 1]$, then F imbeds X into $[0, 1]^J$

Proof. We begin by defining $F(x) : X \to \mathbb{R}^J$ from the product topology $\mathbb{R}^J$ using our set of functions such that

$$F(x) = (f_1(x), f_2(x), \cdots , f_J(x)).$$

We now want to show that this is an imbedding. First, we may conclude that F is continuous as it is the product of continuous functions. Now we must show that F is one-to-one, that is $F(x) \neq F(y)$ if $x \neq y$. Let $x \neq y$. Then there is some neighborhood U_i of x that does not contain y and there is some indexed function f_i such that $f_i(x)$ is positive and $f_i(y)$ is zero. So $F(x) \neq F(y)$ and F is one-to-one. We may be sure of this from the fact that single point sets are closed. Now we know we may define $F(X) = Z$ to create some continuous bijective function $G : X \to Z$. If we are able to show that G is a homeomorphism then we will have shown that F is an imbedding. We can conclude that this is true if it maps open sets to open sets. That is for any open set $U \subset X$ the image $F(U)$ will be open specifically in Z. To do this we will show that for any point $z_0 \in F(U)$ we may find some open set V such that $z_0 \in V$ and $F^{-1}(V) \subset U$. Let $z_0 \in F(U)$ and let $x_0 = F^{-1}(z_0)$. Now suppose that there is some indexed component $f_j(x_0)$ that is positive and such that $f_j(X \setminus U) = 0$. We know this exists from our definition of the components of $F(x)$. Now we want to be able to choose the set of all elements of X that have a nonzero value for their j^{th} component. We may do this by defining the set W to be

$$W = \pi_j^{-1}((0, \infty)),$$

where π_j is the projection map of the j^{th} component and π_j^{-1} gives the preimage. Now, W is the preimage of an open set and so it is open. Furthermore, we may conclude that $V = W \cap Z$ is open in Z by the subspace topology of Z. Now, V gives us the set of all elements of Z that have a positive j^{th} component. We may use that fact to clearly determine that $z_0 \in V$. It is also true that $V \subset F(U)$ as $f_j(x) > 0$ only when $x \in U$ by our definition of the function f_j.Then we have constructed an open set around the point z_0 that is both open and a subset of $F(U)$. This may be done for any point in U and so we may conclude that $F(U)$ is open. Thus, it must be that F is a homeomorphism and therefore an imbedding of X in $\mathbb{R}^J$. Now suppose that f_α instead maps X into $[0, 1]$ for each α. We know that $[0, 1] \subset \mathbb{R}$ and so F imbeds X into $\mathbb{R}^J$ but each image of the function is restricted to $[0, 1]$ and so the image of F will be restricted to $[0, 1]$ as well. □

With the necessary preliminary material covered we may now consider the definition of our key topic, compactification.

Definition (Compactification)**.** Let X be a completely regular space. A compact Hausdorff space Y containing X is a **compactification** of X if Y is the closure of X in Y. If Y_0 and Y_1 are both compactifications of X where there exists some homeomorphism $h : Y_0 \to Y_1$ where $h(X) = X$ we say that Y_0 and Y_1 are **equivalent**.

For a visual example we consider the spaces X and Y shown in Figure 2. We see that Y is a compact Hausdorff space, and that X is a subset of Y. It is also clear that $\overline{X} = Y$ and so Y is a compactification of X.

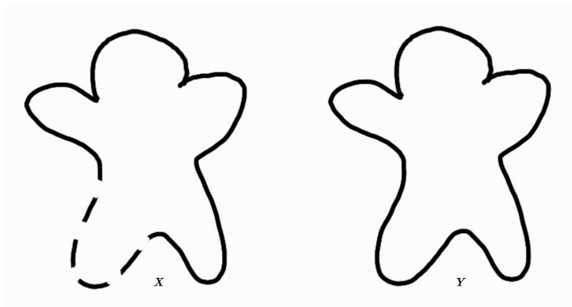

Figure 2: Caption

As noted in our definition, it is possible to have more than one compactification of X. We will now show that compactifications come in many different sizes by constructing several compactifications of the open interval $X = (0, 1)$. First, we will create our smallest possible compactification of X. Suppose we stretch X to be $(-\pi, \pi)$. If we bend the ends of our line towards each other we will have all but one point of a circle. If we add this point in we have a compact circle, specifically the unit circle S^1 that is the one point compactification of X. Figure 3 below gives a visual representation of such a compactification as well as defining the specific imbedding $h : X \to Y$.

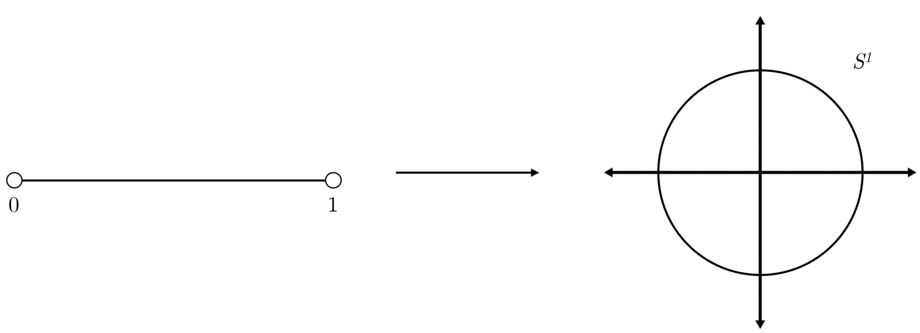

Figure 3: The one point compactification of the open interval $X = (0, 1)$ by the imbedding $h(t) = (\cos(2\pi t)) \times (\sin(2\pi t))$

An obvious compactification of X comes from "gluing in" the pieces missing from some closure X. For example, if our compactification Y is simply the closed interval $[0, 1]$ we need only add in the two missing pieces 0 and 1. Once again this can be seen below in Figure 4.

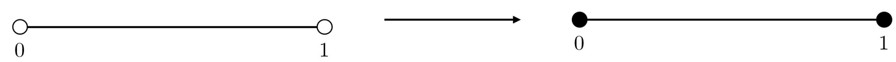

Figure 4: The compactification of $X = (0, 1)$ by completing the closed interval

Now, we will show that if X is completely regular it has a compactification, as Y must be completely regular by the definition. Then it must also be that if we can fit X into any completely regular space Z, we must be able to cut down Z to some Y such that Y is a compactification of X. Formally, this gives us the following lemma.

Lemma 2. Suppose the imbedding $h : X \to Z$ takes the completely regular space X to the compact Hausdorff space Z. Then X has a compactification Y. This compactification may be imbedded in Z by $H : Y \to Z$. Additionally it has the property that if $y \in X$ then $H(y) = h(y)$. The compactification Y induced by h is unique up to equivalence.

So, for example in our one point compactification $Y = S^1$ may be considered the compactification induced by $h : (0, 1) \to S^1$.

Proof. The proof of the existence of a compactification is simple and comes from our understanding of subsets of compact sets. Let h be an imbedding for X in some compact Hausdorff space Z. If $h(X) = X_0$ for some $X_0 \subset Z$ then, by the compactness of Z there exists some compact $Y_0 \subset Z$ such that $\overline{X_0} = Y_0$. Then Y_0 satisfies the necessary conditions to be a compactification of X_0.

Now, we must show that such a Y_0 is unique. Recall that two compactifications Y and Y_0 are equivalent if we may construct some homeomorphism $h : Y \to Y_0$ such that $h(x) = x$ for every $x \in X$. To do this we begin by choosing a set S where $S \cap X = \emptyset$ and where there is some map $k : S \to Y_0 - X_0$ such that k is a bijection. That is, S acts as the boundary of X_0. Using the properties of this map we may define $Y = X \cup S$, with a new bijective map $H : Y \to Y_0$ where

$$H(y) = \begin{cases} h(y) & \text{for } y \in X \\ k(y) & \text{for } y \in S \end{cases}$$

We then give one more property to Y to give it a topology. Suppose that U is a subset of Y. Then U may only be open in Y if and only if $H(U)$ is open in Y_0. The mapping H is a bijection and therefore a homeomorphism. It is also true that $H(y) = h(y)$ for all $y \in X$ and so we can extend the range of H to get an appropriate imbedding $H : Y \to Z$.

Now, we may use this process for any compactification Y_i of X, and so may consider an additional mapping $H_i : Y_i \to Z$. For example, with Y and Y_i we have H and H_1. Now, any such H_i is an extension of h mapping Y onto Y_0. First, note that $X_0 \subseteq H_i(Y_i)$ and as $H_i(Y_i)$ is compact it must be that $\bar{X}_0 \subseteq H_i(Y_i)$ as well. The mapping H_i is continuous and so $H_i(Y_i) \subseteq \bar{X}_0$ meaning $H(Y_i) = \bar{X}_0$. Now if we consider our mappings H and H_i we may say that $H\ -^{-1} \circ H_i$ is a homeomorphism for the sets Y and Y_0 whose result is the identity from X when restricted to X. □

We will now provide a further example of a compactification. This time the compactification induced by the mapping $h : (0,1) \to [-1,1]^2$ defined by $h(x) = x \times \sin\left(\frac{1}{x}\right)$. We see h plotted in Figure 5 below.

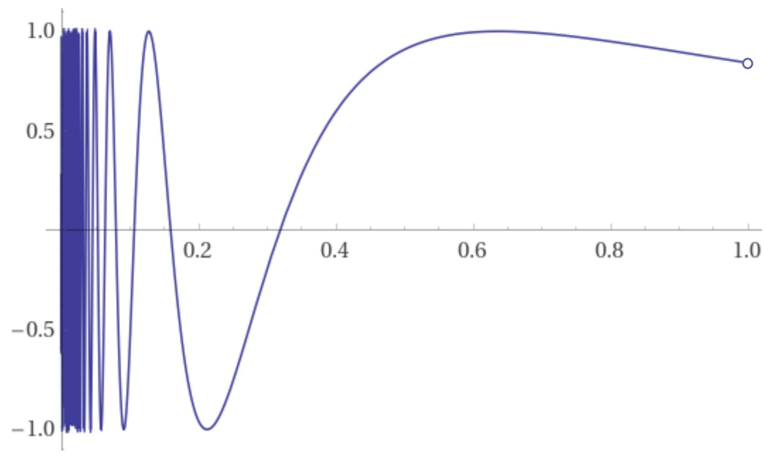

Figure 5: The induced mapping of $h : (0,1) \to [0,1] \times [-1,1]$ given by $h(x) = x \times \sin\left(\frac{1}{x}\right)$ plotted using Wolfram Alpha.

Unlike our compactifications given by Figure 3 and Figure 4 the compactification for Figure 5 is $Y_0 = \overline{h(X)}$ is given by attatching two distinct parts to $h(X)$, specifically,

$$\overline{h(X)} = \{(x, \sin\left(\frac{1}{x}\right)) \in \mathbb{R}^2 \mid 0 < x < 1\} \ \cup\ \{(0,y) \in \mathbb{R}^2 \mid y \in [-1,1]\} \cup \{(1, \sin(1))\} \quad (1)$$

This helps us to see that a compactification can be much larger than previously considered as, unlike our previous compactifications, the inclusion of the line segment $(-1,1)$ adds an uncountably infinite number of points to our $h(X)$. We will find that such compactifications are much more useful for finding continuous extensions of functions on compactifications. That is, if we have some function f that is a continuous and real valued function on X, how can we extend f so that it will be continuous over both X and Y? We will begin by assessing previous examples. First, we once again consider our one point compactification of the interval $(0,1)$. We know that $f : (0,1) \to \mathbb{R}$ must be a bounded function as Y is compact. It must also be true that we obtain the same limit as we get closer and closer to where our two endpoints meet. That is, if we have a bounded continuous function $f : (0,1) \to \mathbb{R}$ we find that f is may be extended to our one point compactification only when it is true that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 1^{-1}} f(x) = c,$$

for some real number c.

When considering our two point compactification from Figure 4 the only restriction is that for the bounded function $f : (0,1) \to \mathbb{R}$ the limits

$$\lim_{x\to 0^+} f(x) \text{ and } \lim_{x\to 1^{-1}} f(x)$$

exist.

For our compactification in Figure 5 we are able to do even more. Obviously a function $f : (0,1) \to \mathbb{R}^2$ has similar dependency on the limits shown above and for example the function $f(x) = x$ is extendable. However, we are able to consider the function $f(x) = \sin(\frac{1}{x})$ and create an appropriate extension for our compactification using a composite map. Let $h : X \to \mathbb{R}^2$ be $h(x) = x \times \sin\left(\frac{1}{x}\right)$. We know from Lemma 2 that there is a unique compactification Y of X and an extension H of h such that $H : Y \to \mathbb{R}^2$ and $H(y) = y \times \sin\left(\frac{1}{y}\right)$ when $y \in X$. If we take the image of H and define a new mapping $\phi : \mathbb{R}^2 \to \mathbb{R}$ we may find the composite map

$$\phi \circ g : Y \xrightarrow{H} \mathbb{R} \times \mathbb{R} \xrightarrow{\phi} \mathbb{R}.$$

If we let ϕ be defined as $\phi(x,y) = y$ then we have created a mapping that shows that the function $f(x) = \sin\left(\frac{1}{x}\right)$ also has an extension for the compactification of X. The important thing to notice here is that when we created our mapping into $\mathbb{R}^2$, both the component functions used in Figure 5 are extendable over our compactification. We will soon find that it is indeed true that if we take some set of n bounded continuous functions $f_1, f_2, \ldots, f_n$ as components in some mapping from X into $\mathbb{R}^n$ then all such functions will have have extensions for the unique compactification Y of X. While unwieldy, we are able to extend on this idea to use brute force to collect all functions that are bounded and continuous over a given X to guarantee a compactification that allows for extensions of all continuous bounded functions of X. This method is at the heart of our titular Stone–Čech compactification. In principle, it is the largest possible compactification of a space X because you are ensuring functions are extendable by the sheer size of your imbedding. Formally, we may define this compactification as

Definition (Stone–Čech compactifion)**.** Let X be a completely regular space. Let $\beta(X)$ be a compactification of X such that for any compact Hausdorff space C and any continuous mapping $f : X \to C$ we have a continuous map $g : \beta(X) \to C$ that acts as a unique extension of f. We call $\beta(X)$ the Stone-Čech compactification

But first, how do we know that this type of extension is possible for all real valued continuous and bounded functions? We have only shown a few examples and must now prove the following Theorem.

Theorem 3. Let X be a completely regular space and let $f : X \to \mathbb{R}$ be any bounded continuous map from X to $\mathbb{R}$. Then there is some compactification Y of X that has the bounded continuous map $g : Y \to \mathbb{R}$ that is an extension of f.

Proof. First, we will need to define an appropriate imbedding and show that it induces a compactification of X. To do this, we denote the collection of all continuous functions with bounded images of X to be $\{f_\alpha\}_{\alpha\in M}$ indexed by M. Now, we want to contain each f_α in a bounded interval which we will denote I_α. A simple way to do this is to set these intervals by,

$$I_\alpha = [\inf f_\alpha(X), \sup f_\alpha(X)]$$

As f_α is bounded for all $\alpha \in M$ this is simply the closure of the image of $f_\alpha(X)$. Further, note that the product of these compact intervals, $\prod_{\alpha\in M} I_\alpha$ creates a new compact space as a result of the Tychonoff's theorem. Now, we know from Lemma 2 that for any imbedding h of a completely regular space X into a compact Hausdorff space such as $\prod_{\alpha\in M} I_\alpha$ has a corresponding compactification Y. To create an appropriate imbedding define the following,

$$h(x) = (f_\alpha(x))_{\alpha\in M}.$$

Then there is a mapping H of the compactification of X induced by h that satisfies the results of Lemma 2, which we will define by

$$H : Y \to \prod I_\alpha.$$

We know that this H is an imbedding from the results of Theorem 1. Now, we consider any function f that is continuous, bounded, and real valued on X and show that this function may be extended to Y using methods similar to that of our previous extension example. By construction, $f \in \{f_\alpha\}_{\alpha\in M}$. Then there is some index β such that $f = f_\beta$. Then we may create some mapping $\phi_\beta : \prod I_\alpha \to I_\beta$. We may compose this function with our imbedding H to create the continuous map

$$\phi_\beta \circ H : Y \xrightarrow{H} \mathbb{R}^M \xrightarrow{\phi_\beta} \mathbb{R}.$$

This is a continuous map from Y into $\mathbb{R}$ and, as $H\mid_X = h$,

$$\phi_\beta(H(x)) = \phi_\beta(h(x)) = \phi_\beta((f_\alpha(x))_{\alpha\in M} = f_\beta(x).$$

Thus, we are able to construct an appropriate extension for any real valued function that is bounded and continuous over X. □

Now that we have shown that such function extensions are possible we must also show that they are unique. To do so, we first will prove a more general Lemma.

Lemma 4. Let A be contained in the completely regular space X and let Z be Hausdorff. If $f : A \to Z$ is continuous then there can be at most one continuous function extension $g : \overline{A} \to Z$ of f.

Proof. Let g and g' both be distinct, continuous extensions of f. Then there must be some x in the boundary of A such that $g(x) \neq g'(x)$. We choose the neighborhood U of $g(x)$ and U' of $g'(x)$ such that $U \cap U' = \emptyset$. Furthermore, by continuity of g and g', we may choose a neighborhood V of x such that $g(V \cap \overline{A})$ and $g'(V \cap \overline{A})$ are subsets of U and U' respectively. Now it is also true that there must be some point $y \in V$ such that $y \in A$ as well. It is clear that for this point, $g(y) \in U$ and $g'(y) \in U'$. It is also true that, by the definition of g and g', it must be that $g(y) = f(y)$ and $g'(y) = f(y)$. This means that $f(y) \in U \cap U'$, which contradicts our earlier statement that they are disjoint. Then g and g' cannot be distinct function extensions. □

We may use the results of Theorem 3 and Lemma 4 to show the existence and uniqueness of our Stone–Čech compactification. Now, we will demonstrate similar results for an extension of *any* continuous function, not just those that are strictly bounded and that this extension is possible for any compact Hausdorff space.

Theorem 5. Let the completely regular space X have compactification Y that behaves as described in Theorem 3. Let C be a compact Hausdorff space. Then for any continuous map $f : X \to C$ we are able to find a unique continuous extension of f in $g : Y \to C$.

Proof. As it is a compact Hausdorff space we may imbed C into the space $[0,1]^M$ for some appropriate M and so for simplicity we may make the assumption that $C \subset [0,1]^M$. We define

$$h(x) = (f_\alpha(x))_{\alpha \in M}.$$

and create the imbedding H of the compactification of X induced by h to be

$$H : Y \to [0,1]^M.$$

Then, we determine the existence of an extension g_α of f_α that sends the compactification Y to the real numbers. We may use such g_α to construct a continuous mapping $g : Y \to [0,1]^M$ where $g(y) = (g_\alpha(y))_{\alpha \in M}$. We now have a mapping that takes Y into the compact space $C \subset \mathbb{R}$. Now, by definition of a compactification $g(Y) = g(\overline{X})$ and the continuity of g ensures that $g(\overline{X}) \subset \overline{g(X)}$. Now, as g is an extension of f it is also true that $\overline{g(X)} = \overline{f(X)}$. The function f has image in C and so $\overline{f(X)} \subset \overline{C}$ but C is compact by definition and so $\overline{C}$ is simply C. Then $g : Y \to C$ and is the necessary function extension. □

Finally, we must show the uniqueness of these compactifications up to equivalency.

Theorem 6. Let Y_0 and Y_1 be compactifications of the completely regular space X such that Y_0 and Y_1 behave as described in Theorem 3. Then the two compactifications are equivalent.

Proof. Define ψ_2 as the continuous function that maps X into its compactification Y_2. We showed in Theorem 5 that any continuous function that maps a completely regular space X into a compact Hausdorff space, such as Y_2 has a function extension for any compactification of X that satisfies the conditions of Theorem 3. We know that Y_1 satisfies this condition by hypothesis and so we may create an extension of ψ_2 defined as

$$f_2 : Y_1 \to Y_2.$$

The same may be said of the inclusion function $\psi_1 : X \to Y_1$ and its extension $f_1 : Y_2 \to Y_1$. Then we may compose our functions such that

$$f_1 \circ f_2 : Y_1 \to Y_2 \to Y_1.$$

So, for every $x \in X$, we have $f_1(f_2(x)) = x$. Then this mapping acts as a continuous extension of the identity mapping $i_X : X \to X$. It must also be true by Theorem 5 that $i_{Y_1} : Y_1 \to Y_1$ is an extension of the identity map i_X as well. We showed with Lemma 4 that i_X may have at most one extension and so $f_1 \circ f_2 = i_{Y1}$ and similar logic may be applied to the function $f_2 \circ f_1$ and the identity map of Y_2. Then f_1 and f_2 are homeomorphisms and therefore Y_1 and Y_2 are equivalent. □

With that, we have shown the existence of the Stone–Čech compactification and proven its key properties.

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